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On the influence of longitudinal diffusion in time-dependent convective–diffusive systems

By H. K. KUIKEN
Philips Research Laboratories, P.O. Box 80000, 5600 JA Eindhoven, The Netherlands

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A model problem is solved mathematically in order to clarify how an essential singularity, which is an integral part of the boundary-layer solution of a time-dependent convective–diffusive system, is removed by the inclusion of the effect of longitudinal diffusion. The model problem involves a uniform velocity field along a plane boundary at which boundary conditions of a mixed type are prescribed. The problem is solved by means of a method involving the Laplace transform and the Wiener–Hopf technique. An exact solution is presented.

Special attention is given to an asymptotic solution that is valid for large values of the dimensionless time. It is shown that the large-time asymptote and the naïve boundary-layer solution are close approximations of one another, except in the neighbourhood of the location where the latter is singular. Around this point the present solution provides an interlayer which matches smoothly the purely time-dependent Rayleigh-like and the stationary components of the boundary-layer approximation.

1. Introduction

In convective–diffusive systems the influence of diffusion in the flow direction is usually excluded from consideration. The reason for this is that diffusion coefficients of the properties or species transported by the flowing medium are usually extremely small, so that even very low velocities are sufficient to render convection the dominant means of transport. Of course, the influence of diffusion cannot in general be disregarded in directions that are more or less normal to the flow, as the spreading out of the field is to be attributed to this effect.

In his pioneering study on the dispersion of soluble matter in a fluid medium flowing through a tube, Taylor (1953) showed that the spreading out of an initial disturbance is due to the combined effects of axial convection and radial diffusion. It was found that axial dispersion was governed by an apparent diffusion coefficient much larger than the actual diffusion coefficient. This showed that axial diffusion, i.e. diffusion in the direction of the flow, is only a lower-order effect. Later studies by Aris (1956) and Lighthill (1966) provided further insight. The subject has prompted a great deal of mathematical study, the most recent contribution being a paper by Liron & Rubinstein (1984).

Another field where time-dependent convective diffusion has led to a great deal of theoretical work is unsteady viscous boundary-layer flow. Here it is vorticity that is diffusing through and convected by the flowing medium. The literature in this area has now become quite prolific, and we refer to a paper by Riley (1975) and to the book by Telionis (1981) for comprehensive surveys. Here we shall restrict our attention to a line of research that was initiated by Stewartson (1951). He considered the motion in a fluid caused by a suddenly started semi-infinite flat plate that is subsequently moving in its own plane at a velocity U in a direction normal to its edge. Stewartson approximated the full Navier–Stokes system by its boundary-layer analogue. In doing so he neglected the influence of longitudinal (along the plate) diffusion of vorticity. As a result, at any instant following the initial disturbance the influence of the leading-edge disturbance cannot have travelled further than what is allowed by the maximum velocity in the system, which is U. Therefore, the flow beyond the position $x = Ut$ (t is the time), as measured from the leading edge, is unaware of this leading edge. The plate might as well have been infinite in all directions, and the flow is given by the simple Rayleigh solution, which is independent of the longitudinal coordinate. In between the leading edge and the position $x = Ut$ the influence of the leading-edge disturbance is noticeable and here the flow does depend upon the longitudinal coordinate. When time tends to infinity the flow field approaches that described by Blasius.

Stewartson was particularly interested in what happens in the neighbourhood of the moving point $x = Ut$ to see how the two solutions matched. In a later study (1973) he showed that the matching of the flows in the two regions requires the introduction of a many-layered flow structure around $x = Ut$. The review paper by Riley (1975) gives a graphic presentation of the pertinent flow field. Dennis (1972) approached the problem by numerical means and found good agreement between his results and those of Stewartson. An important conclusion of Stewartson's research is that the boundary-layer solution is dominated by an essential singularity in the neighbourhood of $x = Ut$.

As we stipulated earlier, the above-mentioned research was carried out within the bounds set by boundary-layer theory. One might wonder how the inclusion of longitudinal diffusion would alter the flow. Clearly, if the boundary-layer approximations are relevant, the overall picture will not change a great deal. However, there is reason to believe that some of the more extreme properties of the boundary-layer solution, particularly the essential singularity near $x = Ut$, might be alleviated to some extent. Indeed, the full system is elliptic as far as all spatial coordinates are concerned, and parabolic in its relation to time. Such systems are known to smooth out any irregularities that might exist at a given instant of time. The occurrence of permanent essential singularities anywhere in the field is not expected. Smith (1970) made similar comments. However, since then this aspect of the problem does not seem to have attracted sufficient attention.

Considering the great complexity of the work reported in Stewartson's papers, we do not think it is feasible yet to investigate the influence of longitudinal diffusion in the problem considered by him. However, we might set ourselves a simpler goal by first attacking less complicated problems that still possess most of the characteristics of the original, the most important of these being the singularity inherent in the boundary-layer approximation. Once the role played by longitudinal diffusion in removing this singularity has been fully understood in these simpler examples, we may hope to be able to tackle the more complicated problem with some measure of success.

2. The model problem and its boundary-layer analogue

Let us consider the region $-\infty < x < \infty$, $y > 0$ as defined in the Cartesian coordinate system (x, y). The region moves as a solid body in the direction of positive x, with velocity U. In this region we shall consider diffusion of some quantity, for instance heat or a dissolved species, whose intensity (concentration) is given by c. The diffusion field is created owing to boundary conditions prescribed at the stationary boundary $y = 0$.

At an initial time $t = 0$ the value of c is equal to zero everywhere. At any subsequent time the value of c is equal to $c_0 \neq 0$ on that part of the boundary $y = 0$ for which $x > 0$. The remaining part of the boundary $y = 0$ remains passive at all times. Denoting the diffusion coefficient of the quantity c by D, we introduce a characteristic length and a characteristic time:

$$l = \frac{D}{U}, \quad \tau = \frac{D}{U^2}. \tag{1}$$

Introducing dimensionless coordinates with the aid of (1) and rendering c dimensionless by means of the reference concentration c_0, we may define the problem in mathematical terms as follows:

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \quad (-\infty < x < \infty, \ y > 0), \tag{2}$$

$$c = 0 \quad \text{at} \quad t = 0 \quad (-\infty < x < \infty, \ y \geqslant 0), \tag{3}$$

$$t > 0 \quad \begin{cases} \dfrac{\partial c}{\partial y} = 0 \quad (y = 0, \ x < 0), & (4) \\ c = 1 \quad (y = 0, \ x > 0). & (5) \end{cases}$$

Moreover

$$c \to 0 \quad \text{if} \quad x \to -\infty \quad \text{or} \quad y \to \infty \tag{6}$$

and

$$\frac{\partial c}{\partial x} \to 0 \quad \text{if} \quad x \to \infty. \tag{7}$$

For reasons of simplicity we use the original symbols for the dimensionless quantities. An Oseen-like system such as the one presented above was considered by Carrier (1959),† who used the problem as an example to illustrate a method for obtaining approximate solutions in Wiener–Hopf problems. The complete solution to the problem was not presented in that paper, but the approximate results revealed many of the properties that we shall bring out in more detail here.

Before we solve this problem, it will be instructive to study first its boundary-layer analogue. The term $\partial^2 c/\partial x^2$ is then omitted from (2). Since the governing equation is now parabolic in both t and x, the solution remains $c = 0$ in the region where both $x < 0$ and $y \geqslant 0$. Consequently, condition (4) should be replaced by $c = 0$ at $x = 0$ $(y > 0)$. The solution can easily be obtained by the application of the Laplace transform with respect to both t and x. Using another method, Stewartson (1951) obtained the result

$$c = \begin{cases} \operatorname{erfc}\left(\dfrac{y}{2x^{\frac{1}{2}}}\right) \quad \text{if} \quad 0 \leqslant x < t, & (8) \\[2ex] \operatorname{erfc}\left(\dfrac{y}{2t^{\frac{1}{2}}}\right) \quad \text{if} \quad x > t \geqslant 0. & (9) \end{cases}$$

From the definition of c as given by (8) and (9) it is clear that all along the moving line $x = t$ the solution displays an essential singularity. Indeed, although c itself is

† This reference was brought to the author's attention by the editor only after the first version of the paper had been submitted.

continuous at $x = t$, all the higher derivatives are discontinuous and it does not seem possible to obtain the solution for $x > t$ from that defined for $x < t$ by means of analytic continuation. As was remarked by Riley (1975), the absence of longitudinal diffusion leads to a hyperbolic character of the governing equation for the combined coordinates x and t. Any discontinuities in the field give rise to waves in the (x, t)-continuum which travel at finite speed. The wavefronts constitute natural boundaries that separate entirely different solutions.

It is the purpose of this paper to show how the inclusion of the longitudinal-diffusion term in (2) will eliminate this singular behaviour. Indeed, the hyperbolic character of the boundary-layer analogue in the (x, t)-continuum is then replaced by a parabolic one. It was again Riley (1975) who remarked that now all points in the field are instantly aware of the presence of any discontinuities that may arise at any particular moment. In other words, if at a given time the c-field is described by an analytic function throughout, a discontinuity applied somewhere in the field at that instant will cause a contribution to the existing field which is again analytic everywhere during all subsequent times. Owing to the linearity of the present system, the two solutions may merely be added to obtain the full solution. In nonlinear cases an intricate interplay will develop between the existing field and the new contribution, but there is no reason to expect the result to be anything but analytic everywhere.

For reasons of completeness we have to dwell a moment on the question of whether or not it is at all realistic to consider a boundary-layer analogue of the system (2)–(7). Indeed, as the system is presented it would seem that there are no small parameters in the problem definition. However, this is merely a matter of appearances. From (9) it is clear that dropping the longitudinal-diffusion term is obviously correct in the region $x > t$. It is also justified to do so in the region $x < t$ if $|\partial^2 c/\partial x^2| \ll |\partial^2 c/\partial y^2|$ there. Applying this boundary-layer prerequisite to (8), we find that the condition

$$y \ll x \tag{10}$$

must be satisfied in the region considered. Moreover, since the erfc function decays exponentially, when its argument assumes higher and higher values, we require that

$$y = O(x^{\frac{1}{2}}). \tag{11}$$

This condition together with that of (10) leads to

$$x \gg 1. \tag{12}$$

Now we must remember that x, as it is used in (12), is dimensionless. Referring to (1) we conclude that (12) implies that the boundary layer is a good approximation at distances from the leading edge that are much larger than l, except in the neighbourhood of the singularity at $x = t$.

To conclude this section we remark that the full solution to the system (2)–(7) is readily obtained when $t = \infty$. The solution becomes stationary. It is most esaily derived by means of parabolic coordinates, as the problem becomes separable. The solution is

$$c = \operatorname{erfc}\left\{\frac{((x^2+y^2)^{\frac{1}{2}}-x)^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right\}, \tag{13}$$

the correctness of which can easily be checked by substitution in the system (2)–(7). When (10) and (12) are valid, (13) is seen to approximate (8).

It is far more difficult to obtain the solution to the full problem when t is finite. This will be the subject of the next sections.

3. The full problem

Upon the application of the Laplace transform

$$\tilde{c} = \int_0^\infty e^{-st}\, c(x, y, t)\, dt \tag{14}$$

to the system of equations and boundary conditions (2)–(7) and the subsequent introduction of

$$\theta(x, y, s) = e^{-\frac{1}{2}x}\, \tilde{c}(x, y, s), \tag{15}$$

we obtain the following set of equations and boundary conditions:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - (s + \tfrac{1}{4})\,\theta = 0, \tag{16}$$

$$\frac{\partial\theta}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad (x < 0), \tag{17}$$

$$\theta = \frac{1}{s}\, e^{-\frac{1}{2}x} \quad \text{at} \quad y = 0 \quad (x > 0), \tag{18}$$

$$\theta \to 0 \quad \text{if} \quad x \to -\infty \quad \text{or} \quad y \to \infty \quad \text{or} \quad x \to \infty. \tag{19}$$

Strictly speaking the condition for $x \to -\infty$ does not follow directly from the original problem formulation, because of the transformation (15). Its validity has to be checked *a posteriori*. However, (13) offers a strong indication of its correctness. Indeed, if $x \to -\infty$, (13) shows that $c \sim \operatorname{erfc}(|x|)$. If this behaviour is substituted in (15) it follows that θ tends to zero in an exponential fashion when $x \to -\infty$ and $t \to \infty$.

The mixed boundary-value problem defined by (16)–(19) can be solved by means of the Wiener–Hopf technique (Noble 1958; Carrier 1959). To that end we introduce the auxiliary functions $g(x)$ and $h(x)$ as follows:

$$g(x, s) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \dfrac{\partial\theta}{\partial y}(x, 0, s) & \text{if} \quad x > 0, \end{cases} \tag{20}$$

$$h(x, s) = \begin{cases} \theta(x, 0, s) & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x > 0. \end{cases} \tag{21}$$

The Fourier transform is now brought into play and we define

$$\vartheta(\omega, y, s) = \int_{-\infty}^{\infty} e^{i\omega x}\, \theta(x, y, s)\, dx = \mathscr{F}_{x\to\omega}(\theta) \tag{22}$$

together with

$$G_+(\omega, s) = \int_{-\infty}^{\infty} e^{i\omega x}\, g\, dx = \int_0^\infty e^{i\omega x}\, g\, dx \tag{23}$$

and

$$H_-(\omega, s) = \int_{-\infty}^{\infty} e^{i\omega x}\, h\, dx = \int_{-\infty}^{0} e^{i\omega x}\, h\, dx. \tag{24}$$

The significance of the plus and minus subscripts in the description of G_+ and H_- is obvious: a plus function is analytic in an upper half-plane $\operatorname{Im}(\omega) > \tau_1$ and a minus function is analytic in a lower half-plane $\operatorname{Im}(\omega) < \tau_2$, where τ_1 and τ_2 are real numbers. The Wiener–Hopf (W–H) technique will work if τ_2 is not smaller, and preferably larger, than τ_1 so that the respective planes of validity overlap. With the W–H technique the usual procedure is to assume such an overlap region to exist beforehand and then to justify this assumption *a posteriori*. However, it may easily be made clear why an overlap region should indeed exist here. Owing to the transformation (15) the function θ decays as $\exp(-\frac{1}{2}x)$ when $x \to \infty$. Since, for finite t, the solution for c is given approximately by (9), we also have from (15) that $\partial\theta/\partial y$ decays as $\exp(-\frac{1}{2}x)$ when $x \to \infty$. From the definition of g given by (20) this implies, together with (23), that G_+ is analytic in $\mathrm{Im}(\omega) > -\frac{1}{2}$, i.e. $\tau_1 = -\frac{1}{2}$. Since h, as defined by (21), is finite, we find that H_- is analytic at least in the region $\mathrm{Im}(\omega) < 0$, so that τ_2 is not smaller than 0, proving that there is indeed a non-vanishing strip $\tau_1 < \mathrm{Im}(\omega) < \tau_2$ where both G_+ and H_- are analytic. Upon completion of the analysis we can even prove that $\tau_2 = \frac{1}{2}$.

We shall now assume for the moment that s is real and positive. Although it is true that this parameter may assume complex values, this condition does not restrict the validity of the final solution. On the contrary, once we have proved that a certain solution is valid on a continuous range of s-values restricted to a line, analytic continuation can be invoked to render that same solution valid for all complex values of s, except possibly at isolated singular points.

When (22)–(24) are applied to (16)–(19) we find

$$\frac{\partial^2\vartheta}{\partial y^2} - (\omega^2 + s + \tfrac{1}{4})\,\vartheta = 0, \tag{25}$$

$$\frac{\partial\vartheta}{\partial y}(\omega, 0, s) = G_+, \quad \vartheta(\omega, 0, s) = H_- + \frac{\mathrm{i}}{s(\omega + \frac{1}{2}\mathrm{i})}, \tag{26}$$

$$\vartheta \to 0 \quad \text{if} \quad y \to \infty. \tag{27}$$

The general solution satisfying (25) and (27) is

$$\vartheta = A(\omega, s)\exp[-y(\omega^2 + s + \tfrac{1}{4})^{\frac{1}{2}}]. \tag{28}$$

At this point it is necessary to define exactly the function $(\omega^2 + s + \frac{1}{4})^{\frac{1}{2}}$ in the complex ω-plane. This function has two zeros, at $\omega = -\mathrm{i}(s + \frac{1}{4})^{\frac{1}{2}}$ and at $\omega = \mathrm{i}(s + \frac{1}{4})^{\frac{1}{2}}$, which are both on the imaginary axis, since s is real. The ω-plane is now assumed to be cut along the imaginary axis from $\mathrm{i}(s + \frac{1}{4})^{\frac{1}{2}}$ to $\mathrm{i}\infty$ and from $-\mathrm{i}(s + \frac{1}{4})^{\frac{1}{2}}$ to $-\mathrm{i}\infty$. The particular branch of the function $(\omega^2 + s + \frac{1}{4})^{\frac{1}{2}}$ is selected by the following conditions:

$$-\tfrac{3}{2}\pi < \arg[\omega - \mathrm{i}(s + \tfrac{1}{4})^{\frac{1}{2}}] < \tfrac{1}{2}\pi, \tag{29}$$

$$-\tfrac{1}{2}\pi < \arg[\omega + \mathrm{i}(s + \tfrac{1}{4})^{\frac{1}{2}}] < \tfrac{3}{2}\pi. \tag{30}$$

These definitions ensure that $\mathrm{Re}\,(\omega^2 + s + \frac{1}{4})^{\frac{1}{2}} > 0$ outside the cuts, so that the function given by (28) has indeed the behaviour prescribed by (27).

If we demand that (28) shall satisfy the two conditions of (26) we find

$$G_+ = -A(\omega^2 + s + \tfrac{1}{4})^{\frac{1}{2}}, \tag{31}$$

$$A = H_- + \frac{\mathrm{i}}{s(\omega + \frac{1}{2}\mathrm{i})}. \tag{32}$$

Eliminating the function A we obtain a functional equation for G_+ and H_-:

$$\frac{G_+}{(\omega^2 + s + \frac{1}{4})^{\frac{1}{2}}} + H_- + \frac{\mathrm{i}}{s(\omega + \frac{1}{2}\mathrm{i})} = 0. \tag{33}$$

To determine the functions G_+ and H_- from (33) we have to factorize this equation in true plus and minus functions. In our case this is quite simple and can be done by inspection. Thus we write

$$\frac{G_+}{[\omega+\mathrm{i}(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}+\frac{\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}[\frac{1}{2}+(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}{s(\omega+\frac{1}{2}\mathrm{i})}$$
$$=-\frac{\mathrm{i}}{s}\frac{[\omega-\mathrm{i}(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}-\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}[\frac{1}{2}+(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}{\omega+\frac{1}{2}\mathrm{i}}-[\omega-\mathrm{i}(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}H_-. \quad (34)$$

The left side of this equation is analytic in the half-plane $\mathrm{Im}\,(\omega) > -\frac{1}{2}$ and the right side is analytic in a half-plane that covers at least $\mathrm{Im}\,(\omega) < 0$. This shows that the two functions, the one on the left side and the one on the right side, are representations of the same entire function. It is well known (Noble 1958) that this function may be determined from the asymptotic behaviour of the two functions when $\omega \to \infty$. Turning first to the left side of (34) we conclude that we must know the behaviour of G_+ when $\omega \to \infty$ in the upper half-plane. This behaviour depends upon that of $\partial\theta/\partial y(x,0,s)$ when $x \downarrow 0$, as can be seen from (20) and (23). If this behaviour is $O(x^{-\frac{1}{2}})$, which we shall verify *a posteriori*, we find $G_+ = O(|\omega|^{-\frac{1}{2}})$ when $\omega \to \infty$. This shows that the left side of (34) tends to zero when $\omega \to \infty$ in the upper half-plane. Assuming next that $h(x)$ tends to unity when $x \uparrow 0$, which is the same as requiring continuity of c at $x = 0$, we find that $H_-(\omega) = O(|\omega|^{-1})$ when $\omega \to \infty$ in the lower half-plane. Thus both the left and the right sides of (34) are zero at infinity. Then Liouville's theorem demands that the entire function shall be equal to zero everywhere. Thus

$$G_+ = -\frac{1}{s}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\,[\tfrac{1}{2}+(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}\frac{[\omega+\mathrm{i}(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}{(\omega+\frac{1}{2}\mathrm{i})}, \quad (35)$$

$$H_- = \frac{\mathrm{i}}{s}\,\frac{-[\omega-\mathrm{i}(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}+\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\,[\frac{1}{2}+(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}{(\omega+\frac{1}{2}\mathrm{i})\,[\omega-\mathrm{i}(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}, \quad (36)$$

$$\vartheta = \frac{\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}}{s}\,\frac{[\frac{1}{2}+(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}\exp[-y(\omega^2+s+\frac{1}{4})^{\frac{1}{2}}]}{(\omega+\frac{1}{2}\mathrm{i})\,[\omega-\mathrm{i}(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}. \quad (37)$$

4. The solution to the full problem

In this section we shall carry out the requisite inverse transformations to determine the solution to the problem defined by (2)–(7). From (22) and (27) we have

$$\theta(x,y,s) = \mathscr{F}^{-1}_{\omega\to x}(\vartheta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega x}\,\vartheta(\omega,y,s)\,\mathrm{d}\omega$$
$$= \mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\frac{[\frac{1}{2}+(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}{2\pi s}\int_{-\infty}^{\infty}\frac{\exp[-\mathrm{i}\omega x-y(\omega^2+s+\frac{1}{4})^{\frac{1}{2}}]}{(\omega+\frac{1}{2}\mathrm{i})\,[\omega-\mathrm{i}(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}\,\mathrm{d}\omega. \quad (38)$$

The path of integration covers the real axis. To carry out the integration in (38) it is advisable to make use of the special definition of the root function that we presented in the paragraph that includes (29) and (30). If x is positive, the integrand in (38) vanishes when $\mathrm{Im}\,(\omega)$ tends to minus infinity. That is why we consider a contour consisting of the interval $(-R, R)$ on the real axis and a semicircle in the lower half-plane interrupted by a loop around the cut on the imaginary axis. Within this contour the integrand of (38) has one pole, viz the one at $\omega = -\frac{1}{2}\mathrm{i}$, whence we have

$$\oint\frac{\exp[-\mathrm{i}\omega x-y(\omega^2+s+\frac{1}{4})^{\frac{1}{2}}]}{(\omega+\frac{1}{2}\mathrm{i})\,[\omega-\mathrm{i}(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}\,\mathrm{d}\omega = -2\pi\frac{\exp[-\frac{1}{2}x-ys^{\frac{1}{2}}+\frac{3}{4}\pi\mathrm{i}]}{[\frac{1}{2}+(s+\frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}, \quad (39)$$

where the contour is understood to have been traversed in the negative (clockwise) direction. When R tends to infinity, the contribution from the semicircle tends to zero so that we are left with contributions from the path along both sides of the cut and a contribution from the real axis, which is the actual integral of (38). The result is

$$\theta = \frac{\exp[-\tfrac{1}{2}x - ys^{\frac{1}{2}}]}{s} - \frac{[\tfrac{1}{2}+(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}{\pi s} \times \exp[-x(s+\tfrac{1}{4})^{\frac{1}{2}}] \int_0^\infty \frac{e^{-\rho x}\sin(y[\rho^2+2\rho(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}})}{[\rho+(s+\tfrac{1}{4})^{\frac{1}{2}}-\tfrac{1}{2}][\rho+2(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}\,d\rho. \quad (40)$$

At this stage it appears that the analysis is simplified if we consider

$$\frac{\partial\tilde{c}}{\partial x} = \frac{\partial}{\partial x}(e^{\frac{1}{2}x}\theta) = \frac{1}{\pi s}\exp(x[\tfrac{1}{2}-(s+\tfrac{1}{4})^{\frac{1}{2}}])[\tfrac{1}{2}+(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}} \times \int_0^\infty \frac{e^{-\rho x}\sin(y[\rho^2+2\rho(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}})}{[\rho+2(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}\,d\rho, \quad (41)$$

where $\tilde{c}$ is defined by (15). The integral appearing in (41) can be solved (Oberhettinger & Badii 1973, formula I7.95) and we have

$$\frac{\partial\tilde{c}}{\partial x} = \frac{[\tfrac{1}{2}+(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}s}\frac{[(x^2+y^2)^{\frac{1}{2}}-x]^{\frac{1}{2}}}{[x^2+y^2]^{\frac{1}{2}}}\exp[\tfrac{1}{2}x-(x^2+y^2)^{\frac{1}{2}}(s+\tfrac{1}{4})^{\frac{1}{2}}]. \quad (42)$$

When x is negative, the integrand of (38) vanishes as $\mathrm{Im}(\omega)$ tends to positive infinity. In that case a contour has to be selected in the upper half of the complex plane, with a suitable loop around the cut. Now there is no singularity within the contour, but it can be shown that the analysis leads to the same result as before. Thus (42) represents the solution for all x and y.

In the same fashion we can derive the inverse Fourier transforms of the functions G_+ and H_- which are given by (35) and (36). These results are

$$\left.\frac{\partial\tilde{c}}{\partial y}\right|_{y=0,\,x>0} = -\frac{[\tfrac{1}{2}+(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}\exp\{-x[(s+\tfrac{1}{4})^{\frac{1}{2}}-\tfrac{1}{2}]\}}{\pi^{\frac{1}{2}}x^{\frac{1}{2}}s} - \frac{\operatorname{erf}(x^{\frac{1}{2}}[(s+\tfrac{1}{4})^{\frac{1}{2}}-\tfrac{1}{2}]^{\frac{1}{2}})}{s^{\frac{1}{2}}}, \quad (43)$$

$$\tilde{c}|_{y=0,\,x<0} = \frac{\operatorname{erfc}(|x|^{\frac{1}{2}}[\tfrac{1}{2}+(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}})}{s}. \quad (44)$$

At this stage in the analysis it may be worthwhile to check some of the properties of the solution as we have derived it so far to see whether they conform to some of our earlier assumptions. First, it is seen that $(\partial\tilde{c}/\partial y)_{y=0}$ has the required singular behaviour when $x\downarrow 0$. To determine the original function c we have to write

$$\frac{\partial c}{\partial x} = \mathscr{L}^{-1}_{s\to t}\left(\frac{\partial\tilde{c}}{\partial x}\right) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\partial\tilde{c}}{\partial x}\,ds, \quad (45)$$

where γ is a positive real number. Therefore $\mathrm{Re}\,(s+\tfrac{1}{4})^{\frac{1}{2}} > \tfrac{1}{2}$ and $\partial\tilde{c}/\partial x$ is seen to vanish when either $y\to\infty$ or $x\to\pm\infty$. This justifies our remarks following (19). Next, we carry out the integration

$$\tilde{c} = \int_{-\infty}^{x}\frac{\partial\tilde{c}}{\partial x}\,dx = \frac{2}{\pi^{\frac{1}{2}}}\frac{[\tfrac{1}{2}+(s+\tfrac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}}{s}\int_{[(x^2+y^2)^{\frac{1}{2}}-x]^{\frac{1}{2}}/2^{\frac{1}{2}}}^{\infty}\exp\left\{-q^2-\left(\frac{y^2}{4q^2}+q^2\right)[(s+\tfrac{1}{4})^{\frac{1}{2}}-\tfrac{1}{2}]\right\}dq, \quad (46)$$

where we have used a suitable change of integration variable. When x tends to infinity, the lower bound tends to zero, and we have

$$\lim_{x\to\infty} \tilde{c} = \frac{1}{s}\,\mathrm{e}^{-ys^{\frac{1}{2}}}, \tag{47}$$

where we have used

$$\int_0^\infty q^{-\frac{1}{2}} \exp\left[-\alpha q - \beta q^{-1}\right] \mathrm{d}q = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} \mathrm{e}^{-2(\alpha\beta)^{\frac{1}{2}}} \quad (\alpha > 0,\ \beta > 0). \tag{48}$$

The Laplace inverse of (47) is easily found to be

$$\lim_{x\to\infty} c = \operatorname{erfc}\left(\frac{y}{2t^{\frac{1}{2}}}\right), \tag{49}$$

showing that the solution approaches the Rayleigh solution (49), as it should. It should also be noted that the lower bound of the integral in (46) is equal to zero when $y = 0$, x being positive. The value of $\tilde{c}$ is then easily shown to be $1/s$, whence $c = 1$, the prescribed boundary condition. Finally, when x is negative, it follows from (46) that $\tilde{c}$ is a symmetric function of y so that $\partial\tilde{c}/\partial y = 0$ when $y = 0$, x being negative. Thus the solution seems to have the expected behaviour in all respects.

The final step in the solution is to find the Laplace inverse of the function defined by (46). Using the result (A 6*b*) of the Appendix we obtain

$$c = \frac{1}{\pi t^{\frac{3}{2}}} \int_{[(x^2+y^2)^{\frac{1}{2}}-x]^{\frac{1}{2}}/2^{\frac{1}{2}}}^\infty \mathrm{e}^{-q^2}\,\mathrm{d}q \int_0^\infty \left(r + q^2 + \frac{y^2}{rq^2}\right) \exp\left[-\frac{1}{4t}\left(r + q^2 + \frac{y^2}{4q^2} - t\right)^2\right] \operatorname{erf}(r^{\frac{1}{2}})\,\mathrm{d}r. \tag{50}$$

It is possible to reduce this expression from a double to a single integral. Since (A 6*b*) and (A 15) of the Appendix represent the same function, the inner integral appearing in (50) may be written as the sum of an integrated part and an integral that may be reduced by partial integration. The final result is

$$\begin{aligned} c = {} & \frac{1}{2\pi}\left(\frac{2}{t}\right)^{\frac{1}{4}} \operatorname{erfc}\left[\frac{((x^2+y^2)^{\frac{1}{2}}-x)^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right] \int_{(x^2+y^2)^{\frac{1}{2}}-t}^\infty \mathrm{e}^{-q^2/4t}\,\bar{K}_{\frac{1}{4}}\left(\frac{(q+2t)^2}{8t}\right)\mathrm{d}q \\ & + \frac{1}{\pi}\left(\frac{2}{t}\right)^{\frac{1}{4}} \int_{[(x^2+y^2)^{\frac{1}{2}}-x]^{\frac{1}{2}}/2^{\frac{1}{2}}}^\infty \left(\frac{2}{\pi^{\frac{1}{2}}}\mathrm{e}^{-q^2} + \left(\frac{y^2}{4q^4} - 1\right) q \operatorname{erfc}(q)\right) \\ & \times \exp\left[-\frac{1}{4t}\left(q^2 + \frac{y^2}{4q^2} - t\right)^2\right] \bar{K}_{\frac{1}{4}}\left(\frac{1}{8t}\left(q^2 + \frac{y^2}{4q^2} + t\right)^2\right)\mathrm{d}q, \end{aligned} \tag{51}$$

where the function $\bar{K}_{\frac{1}{4}}$ is defined by (A 13) of the Appendix. Although this expression seems rather formidable, it offers certain advantages. First, the numerical evaluation will be much quicker than that of (50), since the functions that constitute the integrand can all be obtained by means of efficient subroutines. A double integral can rarely be calculated with the same efficiency.

A second advantage of (51) is that it reveals some important aspects of the structure of the solution c. Indeed, if we keep the spatial coordinates x and y fixed, and let t tend to infinity, the second integral tends to zero in an exponential fashion. This can easily be understood as follows. In the domain of integration, which is fixed, the function $\exp(-(q^2+y^2/4q^2-t)^2/4t)$ can be made exponentially small on any finite range that includes the lower bound by letting t become larger and larger. However, outside any finite range the remaining part of the integrand is exponentially small. As it remains finite, the function $\bar{K}_{\frac{1}{4}}$ does not play a role of importance in this qualitative analysis. For extremely large values of t, the argument of $\tilde{K}_{\frac{1}{4}}$ as it appears in the first integral can be simplified by the application of the asymptotic formula

$$\tilde{K}_{\frac{1}{4}}(\omega) \sim 2^{-\frac{1}{2}}\pi^{+\frac{1}{2}}\omega^{-\frac{1}{4}}[1+O(\omega^{-1})] \quad (\omega \to \infty), \tag{52}$$

where we have used the asymptotic expansion of $K_{\frac{1}{4}}$ valid for large values of the argument (Abramowitz & Stegun 1965, formula 9.7.2). Therefore

$$\begin{aligned} c &\sim 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}} \operatorname{erfc}\left[\frac{((x^2+y^2)^{\frac{1}{2}}-x)^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right] \int_{(x^2+y^2)^{\frac{1}{2}}-t}^{\infty} e^{-q^2/4t} \frac{dq}{(q+2t)^{\frac{1}{2}}} \\ &= \pi^{-\frac{1}{2}} \operatorname{erfc}\left[\frac{((x^2+y^2)^{\frac{1}{2}}-x)^{\frac{1}{2}}}{2^{\frac{1}{2}}}\right] \int_{[(x^2+y^2)^{\frac{1}{2}}-t]/2t^{\frac{1}{2}}}^{\infty} e^{-\sigma^2} \frac{d\sigma}{(1+(\sigma/t^{\frac{1}{2}}))^{\frac{1}{2}}}, \end{aligned} \tag{53}$$

so that c tends to the solution as given by (13), when t tends to infinity, at every fixed point in space.

This state of affairs is changed when t and y are held fixed and x is made to approach infinity. Then, the first integral tends to zero and the solution becomes a function of y and t:

$$\begin{aligned} c = \frac{1}{\pi(2t)^{\frac{1}{2}}} \int_0^{\infty} \left(\frac{2}{\pi^{\frac{1}{2}}} e^{-q^2} + \left(\frac{y^2}{4q^4}-1\right) q \operatorname{erfc}(q)\right)\left(q^2+\frac{y^2}{4q^2}+t\right)^{\frac{1}{2}} \\ \times \exp\left[\frac{1}{4t}\left(\frac{1}{2}\left(q^2+\frac{y^2}{4q^2}+t\right)^2-\left(q^2+\frac{y^2}{4q^2}-t\right)^2\right)\right] \tilde{K}_{\frac{1}{4}}\left[\frac{1}{8t}\left(q^2+\frac{y^2}{4q^2}+t\right)^2\right] dq = \operatorname{erfc}\left(\frac{y}{2t^{\frac{1}{2}}}\right). \end{aligned} \tag{54}$$

The last identity is borne out by (49). However, it would seem a formidable task to prove the second identity of (54) directly, without having any knowledge of the analysis that led from (46) to (51). To be certain, we checked the identity numerically and full agreement was established.

The changeover between the two regimes occurs when $(x^2+y^2)^{\frac{1}{2}} \sim t$. The nature of the solution in this region will be studied in the next section.

5. The large-time asymptote

In this section we shall clarify some aspects of the problem sketched in the introduction, viz the singular matching of the Rayleigh-like and the almost-steady regimes of boundary-layer approximations to time-dependent convective–diffusive systems. As is shown by (12), the notion of a boundary layer is only meaningful if $x \gg 1$. Since the two regimes are separated at $x = t$, we are also required to assume $t \gg 1$. To investigate the large-time asymptote we shall start from (42). From this equation and (A 1) we have

$$\frac{\partial c}{\partial x} = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{[(x^2+y^2)^{\frac{1}{2}}-x]^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{2}}} e^{\frac{1}{2}x} F(t,(x^2+y^2)^{\frac{1}{2}}). \tag{55}$$

When t is much larger than unity we may use the asymptotic expression (A 10) to obtain

$$\begin{aligned} \frac{\partial c}{\partial x} \sim \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{[(x^2+y^2)^{\frac{1}{2}}-x]^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{2}}} \exp\left[\frac{x-(x^2+y^2)^{\frac{1}{2}}}{2}\right]\Bigg[\tfrac{1}{2}\operatorname{erfc}\left(\frac{(x^2+y^2)^{\frac{1}{2}}-t}{2t^{\frac{1}{2}}}\right) \\ + \frac{1}{(\pi t)^{\frac{1}{2}}} \exp\left[-\frac{[(x^2+y^2)^{\frac{1}{2}}-t]^2}{4t}\right]\left\{1-\frac{1}{4t}\frac{2\Omega+1+2\Omega^2(\Omega+1)(x^2+y^2)^{\frac{1}{2}}}{\Omega^3(\Omega+1)^2}+O\left(\frac{1}{t^2}\right)\right\}\Bigg] \\ (t \to \infty), \end{aligned} \tag{56}$$

where

$$\Omega = \left(\frac{t+(x^2+y^2)^{\frac{1}{2}}}{2t}\right)^{\frac{1}{2}}. \tag{57}$$

The term within square brackets is bounded for all values of x and y, and is dominated by the erfc function. Keeping y fixed and letting x vary from $-\infty$ to $+\infty$ we see that the erfc function is at first exponentially small. When x approaches the neighbourhood of $-t$, this function rapidly rises to a value that approximates unity, since $\operatorname{erfc}(-\infty) = 2$. It remains at this level until x is around $+t$, beyond which it drops quite rapidly to exponentially small values. The exponential function outside the square brackets behaves somewhat differently. It is again exponentially small when $x \to -\infty$, its value changes to order unity when x becomes of order unity, and it retains that value as $x \to \infty$. This shows that

$$\frac{\partial c}{\partial x} \sim \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{[(x^2+y^2)^{\frac{1}{2}}-x]^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{2}}} \exp\left[\frac{x-(x^2+y^2)^{\frac{1}{2}}}{2}\right] + \text{exponentially small terms} \tag{58}$$

if $x \ll t$. When (58) is integrated, assuming that $c \to 0$ when $x \to -\infty$, we obtain

$$c \sim \operatorname{erfc}[((x^2+y^2)^{\frac{1}{2}}-x)^{\frac{1}{2}}/2^{\frac{1}{2}}] + \text{exponentially small terms} \quad (-\infty < x \ll t). \tag{59}$$

As soon as the value of x approaches that of t, the rapid change of the term within square brackets in (56) has to be accounted for. To that end we introduce the variables ξ and η as follows:

$$x = t + 2t^{\frac{1}{2}}\xi, \quad y = 2t^{\frac{1}{2}}\eta. \tag{60}$$

If this transformation is substituted in (56), assuming ξ and η fixed and $t \to \infty$, we have

$$\frac{\partial c}{\partial \xi} \sim \frac{2\eta}{(\pi t)^{\frac{1}{2}}} \frac{(1-\xi t^{-\frac{1}{2}}+\ldots)}{(1+2\xi t^{-\frac{1}{2}}+\ldots)} \exp[-\eta^2(1-2\xi t^{-\frac{1}{2}}+\ldots)]\left[\tfrac{1}{2}\operatorname{erfc}(\xi+\eta^2 t^{-\frac{1}{2}}+\ldots) + \frac{1}{(\pi t)^{\frac{1}{2}}}\exp[-(\xi+\ldots)^2]\left\{1-\frac{1}{4\pi^{\frac{1}{2}}t}(t+\ldots)\right\}\right], \tag{61}$$

where the dots stand for terms of higher order in t. Working out the expansion we find

$$\frac{\partial c}{\partial \xi} \sim \frac{2\eta\, e^{-\eta^2}}{(\pi t)^{\frac{1}{2}}}\left[\tfrac{1}{2}\operatorname{erfc}(\xi) + \frac{1}{t^{\frac{1}{2}}}\left\{\xi(\eta^2-\tfrac{1}{2})\operatorname{erfc}(\xi) + (\tfrac{3}{4}-\eta^2)\frac{e^{-\xi^2}}{\pi^{\frac{1}{2}}}\right\} + O\left(\frac{1}{t}\right)\right]. \tag{62}$$

The asymptotic solution in the transition region defined by the coordinates ξ and η may be found by an integration of (62) with respect to ξ. As ξ increases, this solution should approach (49). Therefore

$$c(\xi,\eta) \sim \operatorname{erfc}(\eta) + \frac{1}{(\pi t)^{\frac{1}{2}}}\eta\, e^{-\eta^2}\left(\xi\operatorname{erfc}(\xi) - \frac{1}{\pi^{\frac{1}{2}}}e^{-\xi^2}\right) + \frac{\eta\, e^{-\eta^2}}{\pi^{\frac{1}{2}}t}\left\{\left(\tfrac{3}{2}\xi^2 - \frac{\eta^2}{2} - \xi^2\eta^2\right)\operatorname{erfc}(\xi) + \frac{1}{\pi^{\frac{1}{2}}}(\eta^2-\tfrac{3}{2})\,\xi\, e^{-\xi^2}\right\} + O\left(\frac{1}{T^{\frac{3}{2}}}\right). \tag{63}$$

When $\xi \to \infty$ this function tends rapidly to $\operatorname{erfc}(\eta)$. On the other hand, when $\xi \to -\infty$, c tends to

$$c \sim \operatorname{erfc}(\eta) + \frac{2}{\pi^{\frac{1}{2}}}\xi\eta\, e^{-\eta^2}\frac{1}{t^{\frac{1}{2}}} - \frac{1}{\pi^{\frac{1}{2}}}(3\xi^2-\eta^2-2\xi^2\eta^2)\,\eta\, e^{-\eta^2}\frac{1}{t} + O\left(\frac{1}{t^{\frac{3}{2}}}\right). \tag{64}$$

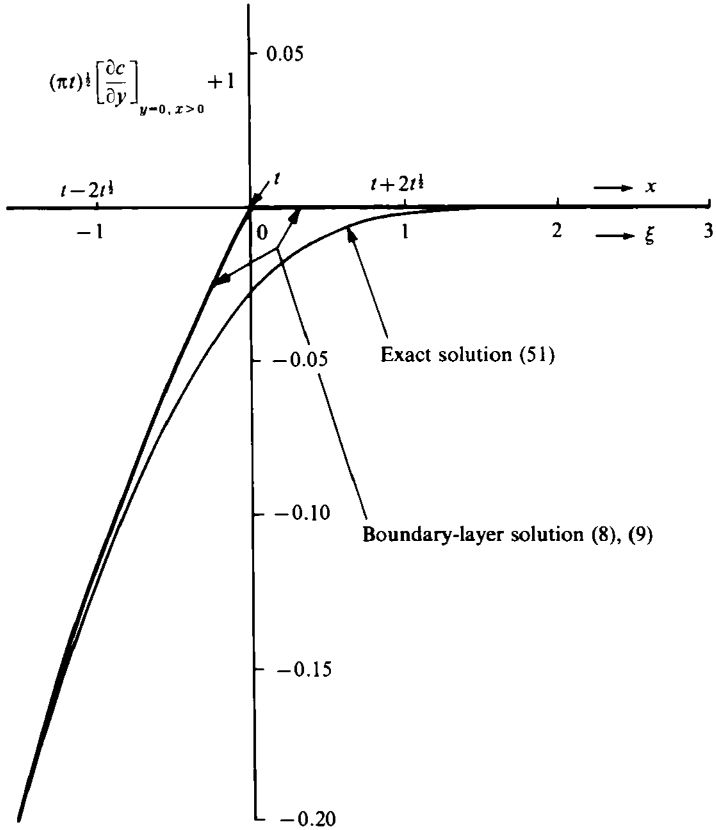

FIGURE 1. The behaviour of the boundary-layer solution and the exact solution in the neighbourhood of $x = t$. The curves presented are related to the mass-transfer function $\partial c/\partial y$ evaluated at $y = 0\,(x \geqslant 0)$. The dimensionless time t has been taken equal to 100.

It should be kept in mind that we are still considering values of x that are $O(t)$. Thus, we have from (60) that $|\xi| \ll t^{\frac{1}{2}}$. However, the exponential and the erfc function in (63) approach their asymptotic values so rapidly that the limit $|\xi| \to \infty$ and the condition $|\xi| \ll t^{\frac{1}{2}}$, although seemingly mutually exclusive, can be applied simultaneously at the cost of exponentially small errors.

When $\xi \to -\infty$, the transition solution (63) should be in agreement with (59). If we substitute (60) in (59) and expand for $t \to \infty$ we find

$$
\begin{aligned}
c &\sim \operatorname{erfc}\left[\eta\left(1-\xi t^{-\frac{1}{2}}+\tfrac{1}{2}(3\xi^2-\eta^2)\,t^{-1}+\tfrac{5}{2}\xi\eta(\eta^2-\xi^2)\,t^{-\frac{3}{2}}+O(t^{-2})\right)\right] \\
&\sim \operatorname{erfc}(\eta)+\frac{1}{(\pi t)^{\frac{1}{2}}}\,\eta\,\mathrm{e}^{-\eta^2}\left[2\xi+(\eta^2-3\xi^2+2\xi^2\eta^2)\,t^{-\frac{1}{2}}\right. \\
&\quad \left.+\left\{\xi(-5\eta^2+2\eta^4)+\xi^3\left(5-\tfrac{20}{3}\eta^2+\tfrac{4}{3}\eta^4\right)\right\}t^{-1}+O(t^{-\frac{3}{2}})\right],
\end{aligned} \tag{65}
$$

showing that there is a smooth matching between the stationary solution (59) and the transition solution (63) up to the order considered (see (64)).

Since the structure of the large-time asymptote is a combination of the two separate solutions (59) and (63) it must be possible to derive it by means of the method of matched asymptotic expansions (Kevorkian & Cole 1981), which proceeds directly from the governing equations. It would be appropriate to study this method here, so that we may get an insight into its possibilities and limitations when applied to problems of the present kind. This is particularly useful when it is impossible to solve such problems exactly. Thus we are fortunate in having an exact solution here, so that we may check the practicability of the method.

As the application of the method of matched asymptotic expansions appears here to be straightforward, we shall not present the details of the derivation. In summary, the governing equation and boundary conditions have to be rewritten in terms of the variables ξ and η defined by (60) and then the asymptotic series may be substituted:

$$c \sim \operatorname{erfc}(\eta) + \sum_{n=1}^{N} t^{-\frac{1}{2}n} c_n(\xi, \eta) \quad (t \to \infty). \tag{66}$$

When matching is restricted to those parts of the asymptotic behaviours that are of algebraic order, the expansion (66) will contain arbitrary constants from the term c_3 onwards. It is possible that this indeterminacy will disappear when matching is extended to terms that are of exponentially small order.

To conclude this section we present a graph which shows how the singular behaviour of the boundary-layer solution at $x = t$ is modified by the influence of longitudinal diffusion. In figure 1 we show the function

$$(\pi t)^{\frac{1}{2}} \left[\frac{\partial c}{\partial y}\right]_{y=0,\, x>0} + 1, \tag{67}$$

which is related to surface mass transfer. The exact solution is seen to display a smooth picture, in contrast to the boundary-layer result.

6. The small-time solution

Although not as interesting as the large-time asymptote, the small-time solution will be briefly discussed here to show a connection with some related work. We may derive the small-time solution by an evaluation of the functions (42)–(44) for large values of s. Restricting ourselves to (43) we have for $s \to \infty$

$$\left[\frac{\partial \bar{c}}{\partial y}\right]_{y=0,\, x>0} \sim \frac{e^{-xs^{\frac{1}{2}}}}{\pi^{\frac{1}{2}} x^{\frac{1}{2}} s^{\frac{3}{4}}} - \frac{\operatorname{erfc}(x^{\frac{1}{2}} s^{\frac{1}{4}})}{s^{\frac{1}{2}}}. \tag{68}$$

This function appeared also in a problem on etching that was recently considered by the present author (Kuiken 1984), which involved the two-dimensional diffusion equation, i.e. the present equation (2) with the convection term $\partial c/\partial x$ omitted. This serves to show that diffusion completely dominates convection in the very first stages of the process.

7. Concluding remarks

By presenting a fairly complete analysis of a model problem, we have succeeded in showing how the inclusion of longitudinal diffusion in a time-dependent convective–diffusive context is instrumental in removing a singularity inherent in the corresponding boundary-layer formulation. The singularity separates a purely time-dependent field, where the limited extent of the pertinent bounding plane has not let itself be felt, from a stationary field which responds instantaneously to the finite extent of this same boundary. It is important to notice that it is not this stationary boundary-layer solution (8), but rather a corresponding solution (13) of the complete stationary equation, that plays a role in the analysis that led to the removal of the singularity. This means that longitudinal diffusion is important not only in the immediate neighbourhood of the singularity at $x = t$: its compounded history from $x = -\infty$ up to $x = t$ is involved in the process.

The knowledge acquired through the study of the present problem may be used to tackle more complicated problems of a related nature. The first of these would seem to be that which involves a less trivial, i.e. a non-uniform, flow field. This obvious choice will be a velocity field with uniform shear and zero speed at the boundary. This would considerably complicate the nature of the boundary-layer singularity, but its solution would also shed more light on the problem that will have to be attacked eventually, namely Stewartson's (1951, 1973).

Appendix

In this Appendix we shall derive the original $F(t,\alpha)$ of the Laplace-transformed function

$$f(s,\alpha) = \mathscr{L}_{t\to s}(F(t,\alpha)) = \frac{[(s+\frac{1}{4})^{\frac{1}{2}}+\frac{1}{2}]^{\frac{1}{2}}}{s}\exp[-\alpha(s+\tfrac{1}{4})^{\frac{1}{2}}] \quad (\alpha \geqslant 0). \tag{A 1}$$

We shall also derive an asymptotic representation of the original that applies for large values of t.

From formula II 3.16 of Oberhettinger & Badii (1973) we have

$$\mathscr{L}^{-1}_{s\to t}((s-\tfrac{1}{2})^{-1}(s+\tfrac{1}{2})^{-\frac{1}{2}}) = e^{\frac{1}{2}t}\operatorname{erf}(t^{\frac{1}{2}}), \tag{A 2}$$

whence

$$\mathscr{L}^{-1}_{s\to t}(e^{-\alpha s}(s-\tfrac{1}{2})^{-1}(s+\tfrac{1}{2})^{-\frac{1}{2}}) = \begin{cases} 0 & \text{if} \quad t<\alpha, \\ e^{\frac{1}{2}(t-\alpha)}\operatorname{erf}[(t-\alpha)^{\frac{1}{2}}] & \text{if} \quad t>\alpha. \end{cases} \tag{A 3}$$

Next we obtain from (A 3) and formula II 1.27 of Oberhettinger & Badii

$$\mathscr{L}^{-1}_{s\to t}(e^{-\alpha s^{\frac{1}{2}}}(s^{\frac{1}{2}}-\tfrac{1}{2})^{-1}(s^{\frac{1}{2}}+\tfrac{1}{2})^{-\frac{1}{2}}) = \frac{1}{2\pi^{\frac{1}{2}}t^{\frac{3}{2}}}\int_\alpha^\infty u\exp\left[-\frac{u^2}{4t}+\frac{u-\alpha}{2}\right]\operatorname{erf}(u^{\frac{1}{2}})\,du. \tag{A 4}$$

Since the function $f(s,\alpha)$ may be written

$$f(s,\alpha) = ((s+\tfrac{1}{4})^{\frac{1}{2}}+\tfrac{1}{2})^{-\frac{1}{2}}((s+\tfrac{1}{4})^{\frac{1}{2}}-\tfrac{1}{2})^{-1}\exp[-\alpha(s+\tfrac{1}{4})^{\frac{1}{2}}], \tag{A 5}$$

we may finally deduce from (A 1), (A 4) and (A 5)

$$F(t,\alpha) = \frac{1}{2\pi^{\frac{1}{2}}t^{\frac{3}{2}}}e^{-\frac{1}{4}t}\int_0^\infty (q+\alpha)\exp\left[\frac{q}{2}-\frac{(q+\alpha)^2}{4t}\right]\operatorname{erf}(q^{\frac{1}{2}})\,dq \tag{A 6a}$$

$$= \frac{e^{-\frac{1}{2}\alpha}}{2\pi^{\frac{1}{2}}t^{\frac{3}{2}}}\int_0^\infty (q+\alpha)\exp\left[-\frac{(q+\alpha-t)^2}{4t}\right]\operatorname{erf}(q^{\frac{1}{2}})\,dq. \tag{A 6b}$$

It is straightforward to derive an asymptotic expression for $F(t,\alpha)$ that is valid in the limit $t\to\infty$. First, we write

$$I(t,\alpha) = \int_0^\infty (q+\alpha)\exp\left[-\frac{(q+\alpha-t)^2}{4t}\right]\operatorname{erf}(q^{\frac{1}{2}})\,dq$$

$$= \pi^{\frac{1}{2}}t^{\frac{3}{2}}\operatorname{erfc}\left(\frac{\alpha-t}{2t^{\frac{1}{2}}}\right)+2t\exp\left[-\frac{(\alpha-t)^2}{4t}\right]-I_1(t,\alpha), \tag{A 7}$$

where

$$I_1(t,\alpha) = \exp\left[-\frac{(\alpha-t)^2}{4t}\right]\int_0^\infty (q+\alpha)\exp\left[-\frac{q^2}{4t}+\frac{t-\alpha}{2t}q\right]\operatorname{erfc}(q^{\frac{1}{2}})\,dq. \tag{A 8}$$

Since $\alpha \geqslant 0$, the integrand of (A 8) decreases at least as fast as $q^{\frac{1}{2}}\,e^{-\frac{1}{2}q}$ when $q \to \infty$, even when $t = \infty$. This is why the first of the two exponential functions appearing in the integrand may be expanded for large values of t, i.e.

$$I_1(t,\alpha) \sim \exp\left[-\frac{(\alpha-t)^2}{4t}\right]\int_0^\infty (q+\alpha)\left(1-\frac{q^2}{4t}+\ldots\right)\exp\left[\frac{t-\alpha}{2t}q\right]\operatorname{erfc}(q^{\frac{1}{2}})\,\mathrm{d}q$$

$$= \exp\left[-\frac{(\alpha-t)^2}{4t}\right]\left\{\frac{2\left(\frac{t+\alpha}{2t}\right)^{\frac{1}{2}}+1+\alpha\frac{t+\alpha}{t}\left(\left(\frac{t+\alpha}{2t}\right)^{\frac{1}{2}}+1\right)}{2\left(\frac{t+\alpha}{2t}\right)^{\frac{3}{2}}\left(\left(\frac{t+\alpha}{2t}\right)^{\frac{1}{2}}+1\right)^2}+O\left(\frac{1}{t}\right)\right\}\quad (t\to\infty). \quad \text{(A 9)}$$

Combining (A 6)–(A 9) we obtain

$$F(t,\alpha) \sim e^{-\frac{1}{2}\alpha}\left[\tfrac{1}{2}\operatorname{erfc}\left(\frac{\alpha-t}{2t^{\frac{1}{2}}}\right)+\frac{1}{\pi^{\frac{1}{2}}t^{\frac{1}{2}}}\exp\left[-\frac{(\alpha-t)^2}{4t}\right]\right.$$
$$\left.\times\left\{1+\frac{1}{4t}\,\frac{2\left(\frac{t+\alpha}{2t}\right)^{\frac{1}{2}}+1+\alpha\frac{t+\alpha}{t}\left(\left(\frac{t+\alpha}{2t}\right)^{\frac{1}{2}}+1\right)}{\left(\frac{t+\alpha}{2t}\right)^{\frac{3}{2}}\left(\left(\frac{t+\alpha}{2t}\right)^{\frac{1}{2}}+1\right)^2}+O\left(\frac{1}{t^2}\right)\right\}\right]\quad (t\to\infty,\ \alpha\geqslant 0). \quad \text{(A 10)}$$

To conclude this Appendix we shall derive a representation of the function $F(t,\alpha)$ that is more suitable for certain purposes. Let us consider the integral $I(t,\alpha)$ defined by (A 7). By writing the term $q+\alpha$ as $(q+\alpha-t)+t$ we may represent I as the sum of two integrals. The first can be reduced by partial integration and we find

$$I(t,\alpha) = \frac{(2t)^{\frac{5}{4}}}{\pi^{\frac{1}{2}}}\exp\left[-\frac{(\alpha-t)^2}{4t}\right]\tilde{K}_{\frac{1}{4}}\left(\frac{(\alpha+t)^2}{8t}\right)+t\int_0^\infty \exp\left[-\frac{(q+\alpha-t)^2}{4t}\right]\operatorname{erf}(q^{\frac{1}{2}})\,\mathrm{d}q, \quad \text{(A 11)}$$

where we have used (Gradshteyn & Ryzhik 1965)

$$\int_0^\infty \exp\left[-q^4-\omega q^2\right]\mathrm{d}q = 2^{-\frac{5}{4}}\tilde{K}_{\frac{1}{4}}\left(\frac{\omega^2}{8}\right). \quad \text{(A 12)}$$

Here $\tilde{K}_{\frac{1}{4}}$ is a shorthand notation for the function

$$\tilde{K}_{\frac{1}{4}}(\cdot) = (\cdot)^{\frac{1}{4}}\,e^{(\cdot)}K_{\frac{1}{4}}(\cdot), \quad \text{(A 13)}$$

where $K_{\frac{1}{4}}$ is a modified Bessel function.

The integral occurring in (A 11), which we denote by $I_2(t,\alpha)$, will be reduced next. Taking the derivative with respect to α we have

$$\frac{\partial I_2}{\partial\alpha} = t\int_0^\infty \operatorname{erf}(q^{\frac{1}{2}})\frac{\partial}{\partial q}\exp\left[-\frac{(q+\alpha-t)^2}{4t}\right]\mathrm{d}q$$
$$= -\frac{2t}{\pi^{\frac{1}{2}}}\exp\left[-\frac{(\alpha-t)^2}{4t}\right]\int_0^\infty \exp\left[-\frac{q^4}{4t}-\frac{\alpha+t}{2t}q^2\right]\mathrm{d}q. \quad \text{(A 14)}$$

Since $\alpha \geqslant 0$, the last integral is given by (A 12). As I_2 tends to zero when α tends to infinity, we may obtain I_2 by a simple integration of (A 14) from α to infinity. If the result is substituted in (A 6*b*) we have, finally,

$$F(t,\alpha) = \frac{1}{\pi}\left(\frac{2}{t}\right)^{\frac{1}{4}}e^{-\frac{1}{2}\alpha}\left\{\exp\left[-\frac{(\alpha-t)^2}{4t}\right]\tilde{K}_{\frac{1}{4}}\left(\frac{(\alpha+t)^2}{8t}\right)\right.$$
$$\left.+\frac{1}{2}\int_\alpha^\infty \exp\left[-\frac{(q-t)^2}{4t}\right]\tilde{K}_{\frac{1}{4}}\left(\frac{(q+t)^2}{8t}\right)\mathrm{d}q\right\}. \quad \text{(A 15)}$$

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